Algorithmic Game Theory
Autumn 2021
Exercise Set 1

> These exercises are **non-graded**. Please submit your your solutions by **email** to
> agt-course@lists.inf.ethz.ch in order to get **feedbacks**,
> before next lecture class (**October 1, 10:00 am**).

Exercise 1:   (1 Points)

Show that best response are not guaranteed to find pure Nash equilibria if they exist. That
is, show a game such that (1) the game has a pure Nash equilibrium, but (2) best response
on this game do not always terminate.

Exercise 2:   (1 Points)

Consider the following weaker form of best response, called *better* response. A strategy $s_i' \in S_i$
is a *better* response to s if $u_i(s) < u_i(s_i', s_{-i})$.

Show that there exist games in which (1) best response terminate always, but (2) better
response do not terminate; that is, there exist a sequence of better response which cycles:

$$s^1, s^2, \ldots, s^k, s^{k+1} = s^1$$

such that each s^{l+1} is obtained from s^l by letting one player changing his/her strategy into
a better response to s^l.

Exercise 3:   (3 Points)

Consider an arbitrary game with *two* players in which best response are guaranteed to ter-
minate from any starting state. Suppose further that in this game best response are always
unique, that is, if $s_i \in S_i$ is a best response to s, then

$$u_i(s_i', s_{-i}) < u_i(s_i^*, s_{-i})$$

for all $s_i' \in S_i$.

Prove that best response reach always an equilibrium in at most $O(m)$ steps, where m is the
number of strategies of each player. (Note that $O(m^2)$ is trivial.)

Discuss how the assumption that best response are unique can be dropped and still obtain
an algorithm which finds a pure Nash equilibrium in $O(m)$ steps.

Exercise 4:   (3 Points)

In this exercise we ask you to prove a key part on the theorem regarding the convergence
time in *Singleton Congestion Games*. Specifically, we have the following (see lecture notes):

1. Given the m resources, sort the set of delay values $V = \{d_r(k) \mid 1 \le r \le m, 1 \le k \le n\}$ in increasing order.

2. Define this alternative new delay functions: $\bar{d}_r(k) :=$ position of $d_r(k)$ in sorted list.

Your exercise is to prove the following:

Suppose $s \to s' = (s'_i, s_{-i})$ is an improvement step for some player i with respect to the original delays. Then $s \to s' = (s'_i, s_{-i})$ is an improvement step for i with respect to the new delays as well.

Here 'improvement step' means exactly better response (exercise above).