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Deadline: Beginning of next lecture

Algorithmic Game Theory

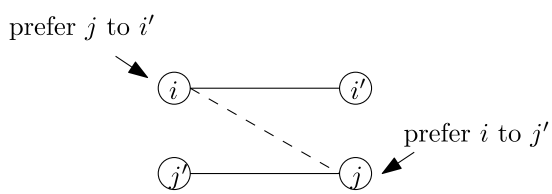
Autumn 2021

Exercise Set 10

These exercises are **non-graded**. Please submit your your solutions via moodle, in order to get **feedbacks**, before the beginning of next lecture (**Dec 13, 10:00 am**). If you cannot use moodle, please send solutions by **email** to agt-course@lists.inf.ethz.ch.

Exercise 1: (2+4 Points)

Recall that in the **stable matching** problem, we have a graph over n nodes (players) and each node has strict preferences over its neighbors. A **stable matching** is a matching such that there are no two players who prefer each other to their matched partners, that is, nothing like this should happen:



In Lecture 10 we have seen an algorithm which computes a stable matching in **bipartite** graphs. Here we consider the **general** version of the problem, that is, we can have any undirected graph.

Your task:

1. Show that in general graphs a stable matching may not exist.

2. Consider the stable matching problem on general graphs restricted to acyclic instances:

Acyclic Instances: There is no cycle of $\ell \geq 3$ players

$$i_1 \to i_2 \to \cdots \to i_\ell \to i_1$$

such that each player prefers the next one over the previous one.

Prove that for acyclic instances the Best-Response Matching Mechanism in the lecture notes converges and is incentive compatible (no player can get matched to a player he/she likes more by misreporting her preferences). Your proof should be based on the "never best response" framework.

Exercise 2: (4+1 Points)

Consider a single-item auction with two bidders. We want to study the relation between **repeated 1^{st}-price auction** and **2^{nd}-price auction**. For this we repeat the definition and introduce a convenient tie breaking rule:

> 1^{st}**-price auction**:
>
> - For $b_1 \geq b_2$ bidder 1 wins and pays b_1;
> - For $b_1 < b_2$ bidder 2 wins and pays b_2.
>
> **Tie breaking rule $\succ_i$:** For any two strategies s and t with $t > s$
>
> $$\text{(at least one above } v_i \Rightarrow \text{prefer the smallest)} : \quad s > v_i \text{ or } t > v_i \implies s \succ_i t \tag{1}$$
>
> $$\text{(both below } v_i \Rightarrow \text{prefer the largest)} : \quad s \leq v_i \text{ and } t \leq v_i \implies t \succ_i s \tag{2}$$

To avoid certain "corner cases" we make the following two assumptions:

- The true valuations v_i are always nonnegative integer, $v_i \in \{0, 1, 2, \ldots\}$;
- Each bidder can make a bid which is a multiple of some small 'minimal increment' $\delta = 1/c$ for integer $c \geq 2$, that is, $b_i \in \{0, \delta, 2\delta, \ldots, 1, 1 + \delta, \ldots, \}$.

Your task:

- Describe **repeated 1^{st}-price auction** as a best-response dynamics and prove that it converges to a unique PNE. In particular, this PNE is "essentially" the same outcome (winner and payments) of **2^{nd}-price auction** on input the true valuations. By "essentially" we mean that in the PNE the winner pays P^{2nd}_{win} or $P^{2nd}_{win} + \delta$, where P^{2nd}_{win} is the price the winner pays in the 2^{nd}-price auction. (Discuss explicitly all cases $v_1 > v_2$, $v_1 = v_2$, and $v_2 > v_1$.)

  (**Hint:** use definition of NBR-solvable game, without the 'clear outcome' part.)

- Suppose we have proven that the above repeated 1^{st}-price auction is an incentive compatible best-response mechanism. Explain how you can deduce from this that 2^{nd}-price auction is truthful (reporting a bid different from the true valuation does not improve the utility of the corresponding player).

Note: In the 1^{st}-price auction (game) above, the utility of a winning bidder i is $v_i - p_i$ where $p_i = p_i(b_1, b_2)$ is the payment computed as above (non-winners have 0 utility).