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Deadline: Beginning of next lecture

Algorithmic Game Theory

Autumn 2021

Exercise Set 12

Your solutions to this exercise sheet will be **graded**. Together with the other three graded exercise sheets, it will account for 30% of your final grade for the course.

You are expected to solve the exercises carefully and then write a nice complete exposition of your solution (preferably using **LaTeX** or similar computer editors – the appearance of your solution will also be part of the grade). You are welcome to discuss the tasks with your fellow students, but you have to hand in **your own** individual write-up. Your write-up should list all collaborators.

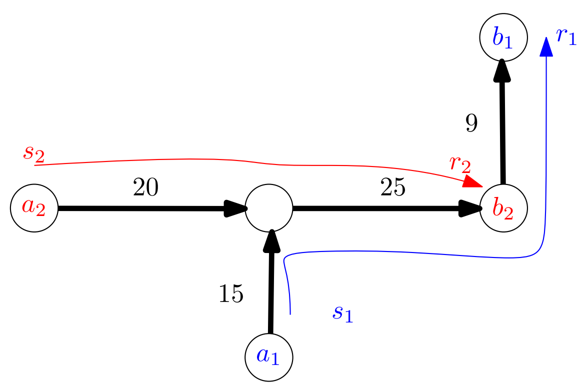
Please submit your solutions via moodle before the beginning of next lecture (**December 17, 10:00 am**). If you cannot use moodle, please send solutions by **email** to agt-course@lists.inf.ethz.ch.

Exercise 1: (4 Points)

Consider TCP games on **general networks** where each edge is a channel of some capacity. Each player sends at a certain rate s_i along a predetermined path, and his/her utility is the rate r_i at which the traffic arrives at the destination (see below for an example).

Show that, if all channels use the same **Strict Priority Queuing** policy (see lecture notes), then the result proved for a single channel extends (the game is NBR-solvable with clear outcome and therefore PIED converges and is incentive compatible).

Here is an example:



In this example, if both players send at rate $s_1 = s_2 = 20$, their respective utility is $r_1 = 9$ and $r_2 = 10$. Indeed, the channel of capacity 15 will allow a rate of 15 from player 1. Therefore, the next edge in the path, the channel of capacity 25, will receive a flow of 15 from player 1 and 20 from player 2. Because of Strict Priority, this channel assigns 15 to player 1 and $25 - 15 = 10$ to player 2. The last edge in the path of player 1 will assign 9 to it, which gives his/her rate r_1.

Note: We assume that the edges are **directed**, and each edge e is a channel of capacity C_e. For any player i, we have a corresponding source-destination pair (a_i, b_i) connected by a fixed path π_i. Player i has a maximum transmission rate M_i, and thus the strategy is the sending rate $s_i \in [0, M_i]$. Each channel e divides its capacity among the players whose path contains this edge as follows. If an edge e in the path of i assigns some rate $r_{i,e}$ to this player, then **the next edge** e' in this path receives a rate $r_{i,e}$. Channel e' applies the Strict Priority to all received rates and determines $r_{i,e'}$. The received rate $r_i(s)$ is the rate that the destination b_i gets from a_i (i.e., the rate that the last edge e in π_i assigns to i).

Exercise 2: (3 Points)

Consider the VCG mechanism for sponsored search presented in the lecture notes:

> The **VCG mechanism for sponsored search** auctions proceeds as follows:
>
> 1. Collect a bid b_i from each agent $i \in \{1, \dots, n\}$.
> 2. Sort bidders such that $b_1 \geq b_2 \geq \dots \geq b_n$. [a]
> 3. For $i = 1, \dots, k$: Assign bidder i to slot i and make him/her pay
>
> $$P_i^{VCG}(b) := \sum_{\ell=i}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot b_{\ell+1} \tag{1}$$
>
> where $\alpha_{k+1} = 0$ represents a "non-existing" slot.
>
> [a] Note: we are renaming bidders according to the bids to avoid overloading notation.

Your task: Show that the mechanism above is indeed a VCG mechanism, as described here (see Lecture 7 for details):

> A **VCG mechanism** is a pair (A, P) such that
>
> - A in an optimal algorithm:
>
> $$SW(A(b), b) = opt_{SW}(b) \qquad \text{for all } b;$$
>
> - P is of the following form:
>
> $$P_i(b) = Q_i(b_{-i}) - \sum_{j \neq i} b_j(A(b))$$
>
> where Q_i is an arbitrary function independent of b_i.

Exercise 3: (3+1 Points)
In this exercise we want to show that the VCG mechanism for sponsored search satisfies the following two conditions:

- **Envy-freeness** meaning that no bidder getting slot s would like to get slot $s+1$ and pay the price of bidder $s+1$, nor slot $s-1$ and pay the price of bidder $s-1$:

$$\alpha_s v_s - P_s^{VCG}(v) \geq \alpha_t v_s - P_t^{VCG}(v) \qquad \text{for } t \in \{s-1, s+1\} \tag{2}$$

- **Voluntary participation** which is the usual condition that truth-telling bidders have non-negative utilities.

Prove that both conditions indeed hold.
Note: As usual for this problem, to avoid overloading the notation, we have renamed the bidders so that $v_1 \geq v_2 \geq \cdots \geq v_s \geq \cdots \geq v_n$.

Exercise 4: (2 Points)
In this exercise we want to show that symmetric pure Nash equilibria do exist. In particular, we want to prove this theorem stated in the lecture notes:

Theorem 10 *There exists always a symmetric pure Nash equilibrium whose revenue is the same as the revenue achieved by VCG on input the true valuations.*

Your task is to prove this theorem: for every valuations v, it is possible to construct bid vector b^{VCG} such that

$$P_s^{VCG}(v) = P_s^{GSP}(b^{VCG})$$

and b^{VCG} is a symmetric pure Nash equilibrium.
Hint: It might be useful to use the simpler characterization of SPE in the lecture notes. It may also help to start from the lowest bidder, the one getting the lowest slot k.

Exercise 5: (3 Points)
Consider the Interns-Hospitals (Stable Matching) problem with **two interns** (1 and 2) and **two hospitals** (A and B) where now hospitals can have **different preferences** over the interns ($\prec_A$ and $\prec_B$).

Prove that for every hospital preferences ($\prec_A$ and $\prec_B$) there is a mechanism which is **obviously strategyproof for the interns**. Explain the following:

1. Whether your mechanism depends on $\prec_A$ and $\prec_B$.
2. Why it returns a stable matching.
3. Why it is indeed obviously strategyproof for the interns.