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October 1, 2021

Deadline: Beginning of next lecture

# Algorithmic Game Theory

## Autumn 2021

## Exercise Set 2

These exercises are **non-graded**. Please submit your your solutions via moodle, in order to get **feedbacks**, before the beginning of next lecture (**October 5, 10:00 am**). If you cannot use moodle, please send solutions by **email** to agt-course@lists.inf.ethz.ch.

**Exercise 1:** (3 Points)

Fair cost sharing games are congestion games with delay functions of the form

$$d_r(x) = c_r/x$$

where $c_r$ is a positive constant. (In these games, $c_r$ represents the cost for building resource $r$, and this cost is shared equally among the players using this resource.)

(a) Show that fair cost sharing games with $n$ players are $(n, 0)$-smooth.

(b) For every $n$, give an example of a fair cost sharing game with $n$ players whose price of anarchy for pure Nash equilibria ($PoA$) is at least $n$.

**Exercise 2:** (3 Points)

An $\epsilon$-Nash equilibrium is a state $s \in S$ such that

$$c_i(s)(1-\epsilon) \leq c_i(s'_i, s_{-i})$$

for all players $i$ and for all $s'_i \in S_i$. Prove that in congestion games with affine delay functions the cost of any $\epsilon$-Nash equilibrium is at most $\frac{5}{2-3\epsilon}$ times the optimal social cost. (The social cost is the sum of all players' costs.)

**Exercise 3:** (3 Points)

In this exercise we consider *load balancing games*: There are $m$ machines of identical speeds. Player $i$ is in charge of one job of weight $w_i > 0$. Every player may choose a machine to process this job; his strategy set is therefore $\{1, \ldots, m\}$. Player $i$'s cost in state $s$ is given as

$$c_i(s) = load_{s_i}(s) := \sum_{i' : s_{i'} = s_i} w_{i'} \ .$$

(Note that this is the load of the machine chosen by $i$, including $w_i$.)

Prove that every pure Nash equilibrium is a local minimum for the function $f$ given by the sum of the *squares of the machine loads*:

$$f(s) = \sum_{\ell=1}^{m} \left(load_\ell(s)\right)^2 .$$

Explain how this implies that a pure Nash equilibrium exists. Does $f$ satisfy the definition of potential game given in the lecture (Lecture 1)?