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Deadline: Beginning of next lecture

# Algorithmic Game Theory

Autumn 2021

Exercise Set 3

Your solutions to this exercise sheet will be **graded**. Together with the other three graded exercise sheets, it will account for 30% of your final grade for the course.

You are expected to solve the exercises carefully and then write a nice complete exposition of your solution (preferably using **LaTeX** or similar computer editors – the appearance of your solution will also be part of the grade). You are welcome to discuss the tasks with your fellow students, but we expect each of you to hand in **your own** individual write-up. Your write-up should list all collaborators.

Please submit your your solutions via moodle, in order to get **feedbacks**, before the beginning of next lecture (**October 15, 10:00 am**). If you cannot use moodle, please send solutions by **email** to agt-course@lists.inf.ethz.ch.

**Exercise 1:** (2+3+1 Points)

In this exercise, we adapt the definition of Price of Anarchy for cost-minimization games, to games with positive utilities in the natural way. Specifically, for $u_i(s)$ being the utility of player $i$ in state $s$, the corresponding **social welfare** is the sum of all players utilities,

$$SW(s) = \sum_i u_i(s) ,$$

and the Price of Anarchy for pure Nash equilibria is

$$PoA_{\mathsf{PNE}} = \frac{\max_{s \in S} SW(s)}{\min_{s \in \mathsf{PNE}} SW(s)} ,$$

where $\mathsf{PNE}$ is set of all pure Nash equilibria in the game, and $S$ the set of all possible states as usual.

We also adapt the definition of $(\lambda, \mu)$-smooth games as follows:

A game as above is called $(\lambda, \mu)$**-smooth** if, for $\lambda > 0$ and $\mu \geq 0$, the inequality

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda SW(s^*) - \mu SW(s)$$

holds for any two states $s^*, s \in S$.

Note: The utilities are always strictly positive, $u_i(s) > 0$ for all possible $s \in S$, where $S = S_1 \times \cdots \times S_n$ are the possible states, with $S_i$ being the strategies available to player $i$.

Your task:

1. Prove the following result:

> **Theorem:** If a game is $(\lambda, \mu)$-smooth, then the Price of Anarchy for pure Nash equilibria satisfies
> $$PoA_{\mathsf{PNE}} \leq \frac{1+\mu}{\lambda} .$$

2. Use the previous theorem to show that the following game has $PoA_{\mathsf{PNE}} \leq n$:

> **Technology game:** There are $n$ players and $m$ different technologies available, where technology $j$ has a quality parameter $\alpha_j > 0$. Each player must adopt one technology, and the more players adopt the same technology, the better it is for them: *If $n_j$ players adopt technology $j$, then each of these players has utility $\alpha_j \cdot n_j$.*

3. Show that the previous bound is tight, that is, $PoA_{\mathsf{PNE}} \geq n$ in some instances of technology games above.

**Exercise 2:** (6 Points)
Nash's Theorem states that every finite strategic game has a mixed Nash equilibrium. Prove this theorem for the special case in which we have only **two players** and each of them has only **two strategies**.

**Note:** Your proof should be elementary and self-contained (in particular, it should not rely on the general proof for arbitrary games, nor it should use the sophisticated arguments in the known proofs for the general case like e.g., Brouwer's fixed point, etc.)

**Exercise 3:** (3 Points)
Consider the class of congestion games with delay functions of the resources of the form $d_r(x) = a_r x + b_r$ with constant $a_r \geq 0$ and $b_r$ any constant (possibly negative) such that $d_r(x) \geq 0$ for all integers $x \geq 1$. Also set $d_r(0) = 0$ as usual.

Show that the price of anarchy for pure Nash equilibria ($PoA_{\mathsf{PNE}}$) can be bigger than 5/2. (Hint: one possibility is to encode delays of the form $d_r(1) = d_r(0) = 0$ and $d_r(2) = M$.)

**Exercise 4:** (6 Points)
Prove that computing pure Nash equilibria in congestion games remains PLS-complete also when we restrict to **affine delay functions**. That is, for $d_r(x) = a_r x + b_r$ with $a_r, b_r \geq 0$. (Note that the condition $a_r \geq 0$ and $b_r \geq 0$ is important.)