Mechanisms Without Money

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[a]These notes are mostly based on older lecture notes by Paul Dütting and Peter Widmayer for this course. All mistakes are mine (Paolo Penna).

Prior lectures: We have seen several mechanism design problems and techniques, all based on the idea of providing or asking **payments (money)** to the players.

This lecture: We consider designing mechanisms (rules) that do **do not use payments (money)**. This applies in all situations where the use of money is undesirable, forbidden, or immoral. As a concrete example consider the "market" for organ donations. Allowing people to pay for organs would be considered unfair as it would give richer people better access to replacement organs. Even worse, it would almost certainly lead to organ trade with all its undesirable repercussions.

> What can we do in situations like this? What can we achieve with mechanisms when we are not allowed to use money?

Our focus will be on ways to guarantee truthfulness, while at the same time constructing good solutions to our problems. We will study this question in three exemplary situations. The allocation of resources such as houses, kidney exchange, and stable matchings or stable marriages.

1 Warm Up: Matching with Goods

Our first problem consists of allocating n items to n players, where each player gets one good. This is a simple **matching** problem. Unlike the classical auction involving money, here we have two important differences:

- There is **no money (payments)** involved.
- Players have a strict **preference** order (list) over the goods.

We assume that players' preferences are represented by a list (vector) P_i of items, from most (top) preferred, to the least (bottom). Each preference P_i is **private** and, as usual, players can possibly report a false preference to the underlying mechanism. Each player cares only about the item he/she gets (like in combinatorial auctions).

Sequential Dictator Mechanism:
Consider the players is some fixed order (say $1, 2, , \dots$):

1. Player 1 gets his/her top ranked (reported) item;
2. Player 2 gets his/her top ranked (reported) item among the remaining (not yet allocated) items;

   $\vdots$

3. Player n gets his/her top ranked (reported) item among the remaining (not yet allocated) items; $\vdots$

Definition 8.1 (incentive-compatible or truthful mechanism)**.** *A mechanism is incentive-compatible (or truthful) if, for every player i, it is a dominant strategy to report his/her true preference list P_i. That is, no matter what the other players report, the allocation computed by the mechanism when i reports a false preference $\tilde{P}_i$ list is not better for i than the one when he/she reports the true one, P_i.*

Theorem 8.2. *The Sequential Dictator Mechanism is incentive compatible (truthful)*

Proof. The proof follows the order of the players considered by the mechanism:

1. Player 1 gets his/her top preference when truth-telling. Reporting a different preference list cannot be better than this, obviously;
2. Player 2 gets his/her top preference among the *remaining* items. No matter the reported preference of player 2, the allocation to player 1 does not change. Hence, player 2 cannot get the item of player 1. Player 2 thus gets the best option among the subset of items that he/she can get – all items except the item a_1 allocated to player 1.
3. Player i cannot get any of the items allocated to players $1, \dots, i-1$. By truth-telling player i gets her top choice in the set of remaining items, and any other reported preference will not give player i any item previously allocated to player $1, 2, \dots, i-1$.

An alternative "inductive" argument is that player $i = 2, \dots, n$ are confronted with a "subgame" in which some items are removed (those allocated previously), and player i is the first considered by the mechanism in this "subgame". □

2 House Allocation

In this resource allocation problem, each of n players has a house, and each player has a strict preference list (a total order) over all n houses. The problem is very similar to the Matching Problem in Section 1 with one important difference:

Each player has his/her own **initial house**

> Can we rearrange the (initial) allocation of the houses so that overall the players are better off?

Naturally, a mechanism cannot force players to accept a new house, but if the mechanism proposes to a player a house that is better for her (i.e., higher on her preference list) than what she currently owns, she will accept. It therefore makes no sense for a mechanism to propose to a player a house that is worse for her than her current house. To get an intuition of how the mechanism should work, consider these two simple situations:

1. If I like my (initial) house more than any other house, I will keep it under all circumstances (i.e., for all other players' preferences).
2. On the other hand, if I like your house best and you mine, we will exchange houses.

The first situation corresponds to a cycle of length one (self loop), and the second situation to a cycle of length two. More generally, we can think of longer cycles along which we could swap houses. This suggests the following algorithm.

Top Trading Cycles Algorithm (TTCA)

0. Initially, all players and all houses remain.

   While players remain, do the following:

1. Let each remaining player point to her favorite (top) remaining house. This describes a directed graph where each vertex has out-degree one.
2. Reallocate as suggested by the directed cycles in the graph (including self loops), and delete the reallocated houses and players. All other players keep their current houses.

The TTC Algorithm has four important and highly desirable properties.

1) Termination The algorithm terminates, because Step 2 always reallocates along at least one cycle. The reason for the existence of a cycle is the fact that each vertex has exactly one outgoing edge, and therefore we can start at any vertex and follow outgoing edges as long as we wish. At some point, a vertex must repeat, and a cycle is found.

2) Weakly improved allocation At the end, each agent has a house that she likes at least as much as her initial house. The reason is that after Step 2 of TTCA, all remaining players keep their houses, and the agents that got a house in fact pointed to this house. If a player points to her own house in an iteration of the loop, she gets it. Other houses can be allocated to a player only earlier, when the player points to a house higher up on her list than her own house.

3) Incentive compatibility TTCA is incentive compatible. The reason for incentive compatibility can be explained inductively. Each player who gets a house in the first iteration of the loop will get her first choice on the preference list, so she has no incentive to lie. Let N_i denote the set of players who get a house in iteration i. We have just seen that for each player in N_1, TTCA is incentive compatible.

Observe that no player from among N_i points to a player j from among N_{i+1}, by contradiction: If it would, then player j would have to be part of the same cycle, and therefore would be in N_i. Therefore, a player in N_{i+1} cannot become part of N_i (or N_k for $k < i$) by lying, because lying only changes its outgoing arc, but not its incoming arc. Since player j gets her favorite house outside of $N_1 \cup \ldots \cup N_i$ and cannot get a house in $N_1 \cup \ldots \cup N_i$, she has no incentive to lie.

While these properties may appear strong at first sight, they are trivial to achieve:

Remark 8.3 (Trivial solution exists)**.** *The "do nothing" mechanism which gives every player her initial house satisfies termination, weak improvement, and truthfulness.*

This trivial mechanism can be beaten easily in the "I like your house better and you mine" scenario (Item 2 at page 3): We will simply exchange houses by "ourselves". This observation suggests the following requirement: A mechanism should make it impossible for a subset of players to exchange houses among themselves and be better off.

An allocation a **core allocation** if no coalition (i.e., subset) of players can reallocate their initial houses and make at least one of its embers better off and, at the same time, none of its members strictly worse. Then we say that there is no **blocking coalition**.

4) Unique core allocation. TTCA finds the unique allocation in the core. The proof that the TTCA allocation is in the core is by contradiction. Assume there exists a blocking coalition S, a subset of all players. Let i be the first iteration in which a player j in S gets a house, i.e., the smallest i for which $N_i \cap S \neq \emptyset$. Note that player j gets her favorite house outside $N_1 \cup \ldots \cup N_{i-1}$. Since no player of S belongs to $N_1 \cup \ldots \cup N_{i-1}$, no reallocation within S can make j better off, a contradiction. To show that the allocation in the core is unique, we observe that all players in N_1 get their top choices. Hence, these players must get their top choices also in any core allocation, otherwise they would contain a blocking coalition formed by those players in N_1 who did not get their top choices. Similarly, all players in N_2 must get their houses as allocated by TTCA also in any core allocation. By induction, the same holds for all players.

3 Kidney Exchange

One might ask for which practical settings the house allocation problem arises. Kidney exchange appears to be similar to house allocation. Kidney transplantation from a living donor to a patient in need has become a routine procedure. Most often, the willingness of a donor to donate a kidney is limited to spouses, close relatives and friends. But not each kidney is suitable for each patient (blood type and other factors play a role), so the kidney of a willing donor may not be suitable for the patient. That's why kidneys should be exchanged among pairs of patient and donor, to help as many patients as possible.

Each patient-donor pair i is a **player**, with preferences over the other players (potential donors).

If for a set of pairs of patient and donor, each patient has a total preference order over all kidneys (taking into account factors such as blood type, tissue type, and many more factors), the TTCA mechanism can be used to make everybody better off. In practice, we may not use TTCA for kidney exchange:

Problems of TTCA for kidney exhange:

1. Long cycles $\Longrightarrow$ Several simultaneous operations
2. Preference list over all donors "too detailed/not appropriate"

1. The cycles produced by TTCA may be very long. A **long cycle** corresponds to lots of surgeries that should happen at the same time and at the same hospital (rather difficult or even impossible). The condition for all involved surgical operations to happen simultaneously comes from the problem that after a willing donor's spouse got a new kidney, the formerly willing donor may no longer be willing to donate a kidney. This not only gives an unfair free kidney to his spouse, but also prevents the other donor's spouse from taking part in the kidney exchange in the future, since she has no donor to offer.[1]
2. Another reason is that a total preference order over all kidneys may be an overkill, and it may be preferable to simply distinguish suitable kidneys from unsuitable ones.

We thus consider the following setting:

1. We search for **pairs** to be matched (cycles of length two);
2. The **preferences** of each player are "binary" that is, they consist of a list of all (equally good) compatible donors.[a]

[a]Hence, for player i, each donor is either "good=compatible" or "bad = not compatible".

Instead of a directed graph, we consider an undirected graph, where each vertex is a patient-donor pair (player) u, and each edge (u, v) tells that both pairs u and v are interested in an exchange. We look for a matching of maximum cardinality in this exchange graph, where a matching is a subset of the set of edges in which no two edges share an end vertex. Now the goal is to find a maximum cardinality matching in a truthful way, where the private information of a patient-donor pair is the set of acceptable other patient-donor pairs, that is, the set of outgoing arcs in the directed graph that gives rise to the undirected graph.

Definition 8.4 (Incentive compatible kidney exchange)**.** *An algorithm (or mechanism) for kidney exchange is incentive compatible if reporting all outgoing arcs is a dominant strategy for each player (each player's goal is to get matched).*

We will show that the following mechanism achieves this objective.

[1]An opposite "fully altruistic" behavior lead to the longest transplant chain so far – see here for the full story.

Maximum Matching Mechanism

1. Get a bid from each player (patient-donor pair) consisting of the acceptable edge set F_i for player i.
2. Define the set of edges on which the players agree as $E = \{(i,j) \mid (i,j) \in F_i \cap F_j\}$.
3. Return a maximum (cardinality) matching.

In order to prove truthfulness, we need to clarify which of the maximum matchings should be returned in Step 3. Maximum matchings differ in two possible ways:

1. First, in an even length cycle, each of two alternating matchings can be chosen. No matter which one is chosen, the same patient-donor pairs get kidney exchanges, and therefore there is no room for strategizing.
2. Second, for a single node with several incident edges as in a star, any one of these edges might be chosen, which might give rise to strategizing.

One can avoid this by giving priorities to patient-donor pairs initially, according to medical criteria (e.g., how much in need is a patient, how long did she wait already) or circumstantial criteria, and by choosing the edge that matches a highest priority vertex if there is a choice. Let the priorities be reflected in the identities of the vertices. Then, Step 3 becomes:

Priority Matching Mechanism (Cont'd)

3a. Let M_0 be the set of all maximum matchings of the given graph.

3b. Loop for $i = 1, \dots, n$:
// The order of vertices in this loop reflects their priorities
// Serve vertex i as best you can
If some matching in M_{i-1} matches vertex i, then kick out all matchings from M_{i-1} that do not match vertex i, yielding M_i.
Else leave M_{i-1} as is, yielding $M_i = M_{i-1}$.

3c. Return an arbitrary matching from M_n.

Note that since M_0 is nonempty, so is M_n. It is easy to see by induction that the priority matching mechanism is truthful in the sense that no vertex can go from unmatched to matched by reporting a proper subset of its true edge set, for every collection of true edge sets and every ordering of the vertices.

Remark 8.5. *In practice, one may consider the variant in which players are not individual patient-donor pairs, but hospitals with several such pairs. Each hospital aims at maximizing its own patients matched by the algorithm/mechanism. The mechanism represents a national (or even wider) kidney exchange office to which hospitals are supposed to report their patient-donor pairs.*

Unfortunately, the above Priority Matching Mechanism is not incentive-compatible for the hospital setting described above. That is, there exists a situation in which, by not reporting some patient-donor pair, an hospital can increase the number of its patients matched (doing some matchings internally).

4 Stable Matching

The stable matching problem differs from the kidney exchange matching problem in two ways. Preferences are not binary, but each player has a total order over all alternatives, and the result of the matching is required to contain no blocking pair of vertices. Since stable matching is a workhorse for many situations and has been used for decades in settings such as assigning medical school graduates to hospitals, assigning room mates to dormitories, and many more, let us describe it independently of kidney exchange in full detail, at the classical example of matching men and women. We limit ourselves to *bipartite* stable matching, also called *stable marriage*. We are given n men and n women. Each man has a total preference order over all women, and each woman has a total preference order over all men. As an example, consider the set U consisting of three men A, B, and C, and the set V consisting of three women D, E, and F. Assume the preferences of the men are all identical lists D, E, F, where D is liked best and F least. For woman D, the preferences are A, B, C. For woman E, they are B, C, A. For woman F, they are C, A, B. The graph representing the possible matchings is a complete bipartite graph between the vertices in U and those in V.

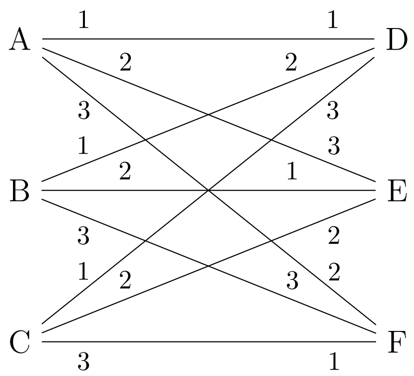


Figure 1: Visualization of the example. Edge labels denote preferences. Labels range from 1 (most preferred) to 3 (least preferred).

A maximum matching in a complete bipartite graph matches every vertex and is therefore called a *perfect matching*. Among all perfect matchings, our goal is to identify one that does not contain a blocking pair, i.e., a pair $u \in U, v \in V$ of vertices that are not matched, but prefer each other to the partners to which they are matched. Obviously, such a pair would make the matching unstable, because u could simply run away with v from their marriages and be better off.

Let us study the classical algorithm to solve this problem, the *Proposal Algorithm* by Gale and Shapley.

Proposal Algorithm

1. Initially, nobody is matched.
2. Loop until all men are matched
   Let man u propose to his favorite woman who has not rejected him yet.
   Let each woman only entertain her best offer so far and reject all others.

Observe that this algorithm is not fully determined, because it is open which man u to choose in an iteration of the loop. Still, one can see a few properties of the algorithm:

1. Over time each man goes through his preference list sequentially, from best to worst.
2. For each woman, the men that she accepts over time (and maybe later rejects) get better over time.
3. At any point in time, each man is matched to at most one woman, and vice versa.
4. The proposal algorithm terminates after at most n^2 iterations. The reason is that each man asks each woman in his list at most once.
5. The proposal algorithm terminates with a perfect matching. Otherwise, a man would have been rejected by all women. But a woman rejects a man only if she has a better man, of which there are only $n-1$. So, only $n-1$ women can end up with a better man than, and one woman would be left without a man, a contradiction.
6. The proposal algorithm terminates with a stable matching. For the sake of contradiction, consider man u not matched to woman v. There are two possible reasons for u not being matched to v: Either u never proposed to v, or u proposed to v at some point, but was rejected by v (either upon proposing or later). In the first case, u must be matched to a woman higher than v on u's list. But then the pair u, v is not blocking. In the second case, v must have rejected u for a man higher on v's list than u, so also in this case, u, v are not a blocking pair.
7. As a corollary, we now know that for every collection of preference lists a stable matching exists.

After observing these properties without having specified how the next man u to propose is chosen, let us now study how we should choose the next man to get the best result. At this point, it is not even clear what the range of possible results is that the proposal algorithm generates, and how good they are for the men and women. For man u, let $h(u)$ denote the best woman u can possibly get, i.e., the highest ranked woman on u's list that is matched to u in any stable matching. Amazingly, each man u gets $h(u)$ in the proposal algorithm, as the following theorem states:

Theorem 8.6. *For every man $u \in U$, the proposal algorithm matches u with $h(u)$.*

This in particular means that it makes no difference which man u is picked next.

Proof. For a pair u, v that is matched by the proposal algorithm, we know that any woman v' whom u prefers to v must have rejected u at some point. We only need to prove that this will always be fine i.e., that whenever a woman v' rejects a man u at some point, no stable matching pairs u and v'. This will imply the theorem. We prove the claim by induction. Initially, no woman rejected any man, so the claim holds. Now consider the rejection of a man u by a woman v'. Note that v' rejects u in favor of a better man u'. Since u' worked from the first woman of his preference list down to v', every woman that u' prefers to v' rejected him already. By the inductive hypothesis that no stable matching matches a man to a woman that rejects him in the proposal algorithm, no stable matching matches u' with a woman whom he prefers to v'. As v' prefers u' to u, and u' prefers v' to any woman he might get in any stable matching, it would be unstable to match u with v', and hence no stable matching pairs u with v'. This concludes the induction. □

Interestingly, the stable matching found by the proposal algorithm turns out to be worst for the women, a fact that we will not prove here. So, the proposal algorithm is really a *m*ale proposal algorithm that gives every man his best woman. It comes therefore as no surprise that it is truthful for the men, but not for the women, when preference lists are private:

Theorem 8.7. *The male proposal algorithm is incentive compatible for the men, but not for the women.*

Proof. No truthfulness for the women can be demonstrated by an example. Assume we are given men D, E, F and women A, B, C (for a change of names). The preferences are as follows: Man D ranks the women B, A, C. Man E ranks the women A, C, B. Man F ranks the women A, B, C. Woman A ranks the men D, F, E. Woman B ranks the men F, D, E. Woman C ranks the men D, F, E. These are the true ranks, and the male proposal algorithm matches D with B, E with C, and F with A. If, however, woman A lies and declares D, E, F instead, she gets man D, a better man for her.

Truthfulness for the men can be seen by contradiction. Assume some man u lies and improves. Let M be the stable matching that the male proposal algorithm produces for true preferences, and let M' be the matching produced for the preference lists in which u lies. Let R be the set of all those men who improve in M' as against M. Let S be the set of women matched to men in R in the matching M'. Let v be the woman that u gets in M'. Since M is stable, we know that v cannot prefer u to the man she got in M, because this would make u, v a blocking pair in M (recall that man u is better off in M' than in M, so he prefers v to the woman he gets in M). In other words, woman v prefers the man she gets in M to u. Now, if v's man in M would not improve in M', he would propose to v in M', and since v prefers him to u, v could not be matched with u in M', a contradiction. Therefore, v's man in M also improves in M', that is, belongs to R. Hence, S is not only the set of women in M' of the men in R, but also the set of women in M of the men in R. In other words, each woman in S is matched to two different men from R in matchings M and M', being better off in M than in M'. We will now show that M' cannot be stable, a contradiction that terminates the proof. We show this by looking at M and focusing on the last proposal of a man from R in the proposal process that leads to M. Call the man who proposes last $u' \in R$. This proposal must be to his woman in M, say v', and v' must accept for this proposal to be last from among men in R. We know already that $v' \in S$. Now observe that every woman in S rejects in M her man in M', because this man prefers her over his woman in M and hence went through his preference list proposing to the rejected woman earlier. Especially, v' must have rejected her man in M' already in the male proposal algorithm producing M, so v'

must have had a man u'' when u' proposed to her, and she must have rejected this man for u'. Because v' only improves as time passes and she accepted u'' after rejecting her man in M', she prefers u'' to her man in M'. Note that in particular, u'' cannot be her man in M', because if it were, u'' would belong to R and would need to propose again, contradicting the assumption that the proposal by u' is the last one from among R. For this reason u'' is known to be outside R. After u'' is rejected by v', he ends up with a woman lower on his list than v'. This, however, makes M' unstable, as we will now show. Woman v' prefers u'' to her man in M', because she accepted u'' after having accepted her man in M'. Man u'' in turn prefers v' over his woman in M, because he got rejected by v' who must therefore be higher on his list. Furthermore, u'' prefers his woman in M over his woman in M', because he does not belong to R. Hence, u'' prefers v' to his woman in M', so these two form a blocking pair, making M' not stable. □

Recommended Literature (lecture notes/books)

- Tim Roughgarden, Lecture Notes for 364A: Algorithmic Game Theory, Lecture #9: Beyond Quasi-Linearity, 2015. (House allocation)
  http://timroughgarden.org/f13/l/l9.pdf

- Tim Roughgarden, Lecture Notes for 364A: Algorithmic Game Theory, Lecture #10: Kidney Exchange and Stable Matching, 2015. (Kidney exchange and stable matching)
  http://timroughgarden.org/f13/l/l10.pdf

- James Schummer and Rakesh Vohra, Algorithmic Game Theory, Chapter 10: Mechanism Design without Money, Cambridge University Press, 2007. (General introduction to the topic)

Further Literature (original results)

- Lloyd S. Shapley and Herbert Scarf, On Cores and Indivisibility. Journal of Mathematical Economics, 1(1):23–28, 1974. (TTC Algorithm)

- Alvin E. Roth, Tayfun Sönmez, and M. Utku Ünver. Kidney Exchange. Quarterly Journal of Economics, 119(2):457–488, 2004. (Original results on kidney exchange)

- David Gale and Llyod S. Shapley, College Admissions and the Stability of Marriage, American Mathematical Monthly, 69: 9–14, 1962. (Man Proposal Algorithm)

- Mehmet Karakaya, Bettina Klaus, Jan Christoph Schlegel. Top Trading Cycles, Consistency, and Acyclic Priorities for House Allocation with Existing Tenants. Journal of Economic Theory Volume 184, 2019.
  (House Allocation variant when some players may not have an initial house)

- Jinpeng Ma. Strategy-Proofness and the Strict Core in a Market with Indivisibilities. International Journal of Game Theory (1994) 23:75-83
  (Core and equivalence with other natural conditions)

Exercises
(during next exercise class - 16.11.2021)

This in-class exercise is on the **correct definition** of **core**.

Def. An house allocation is in the **core** if there is no **blocking coalition**.

The real issue is to define **blocking coalition**...Let's see

We say that a subset C of players is a blocking coalition for allocation a if:

Ver 1 - All improve: Members of C can reallocate the houses they get in a among themselves so that **all** of them are strictly better;

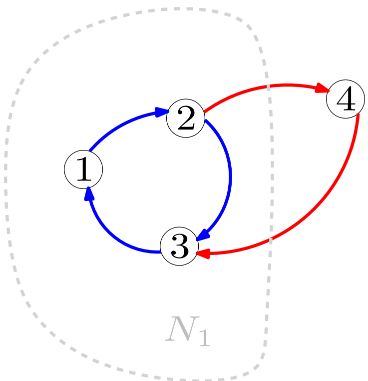
Ver 2 - One improves and none worse: Members of C can reallocate the houses they get in a among themselves so that **at least one** of them is strictly better, and none of them is worse;

Which version is the correct one?

Exercise 1. *The TTC Algorithm seems to "not work" with Ver 1. In particular, the proof that TTCA returns the unique core allocation a is based on this argument:*

> *Any other allocation a' which changes the assignment of allocation a for some players in N_1 makes these players a blocking coalition.*

*It seems that with **Ver 1** above this is not true:*



We could replace edge $2 \to 3$ with $2 \to 4 \to 3$ and get a new allocation a'. Find preferences such that (a) TTCA produces the blue cycle (allocation) and (b) For the "red cycle" allocation a' there is no subset C of players that can all improve if they exchange the houses they get in a'.

Exercise 2. *So, must use **Ver 2** of definition of blocking coalition. Remember we want to show that the (TTCA) core allocation is **unique**. Show that this is also **false**: Give an example where there are **two** allocations which do not contain any blocking coalition according to **Ver 2**. (Hint: two players are enough.)*

The **initial allocation** is important

Each player i has some **initial house** i. We can define "improvement" and "blocking coalition" accordingly:

Ver 3 : Members of C can reallocate their **initial houses** among themselves so that **at least one** of them is strictly better, and **none** of them is **worse**;

Exercise 3. *Check that in the two previous examples there is only one core allocation when defining blocking coalition as in* ***Ver 3****. Go back to the proof of "unique core allocation" of TTCA and discuss how it works.*

You can see the definition of the core (and variants) in e.g. references (Ma, IJGT 1994) and (Karakaya, Klaus, Schlegel, JET 2019) above.