A Note on the Computational Hardness of Evolutionarily Stable Strategies

Overview

- The Tools
  - Game Theory – what the smart thing to do?
  - Complexity – is chess harder then sudoku?
  - Reductions
- The Method
  - From Graph to Strategy
- Claims and Proofs

Game Theory

- mathematics of strategic interaction

Payoff Matrix – The Prisoner's Dilemma

Prisoner 1

Prisoner 2

	Cooperate	Stay Silent
Cooperate	-3 / -3	-4 / 0
Stay Silent	0 / -4	-1 / -1

Nash Equilibrium

Prisoner 1

Prisoner 2

	Cooperate	Stay Silent
Cooperate	-3 / -3	-4 / 0
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Symmetric Nash Equilibrium: *"Best response to itself"*

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Symmetric Nash Equilibrium: *"Best response to itself"*

Rock-Paper-Scissors

	Rock	Paper	Scissors
Rock	0/0	1/-1	-1/1
Paper	-1/1	0/0	1/-1
Scissors	1/-1	-1/1	0/0

Rock-Paper-Scissors

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Scissors	1/-1	-1/1	0/0

Solution? Mixed Strategies!

Mixed Strategy

Pick each option with a certain probability

- Rock 50%, Scissors 25%, Paper 25%
- Rock 33%, Scissors 33%, Paper 33%

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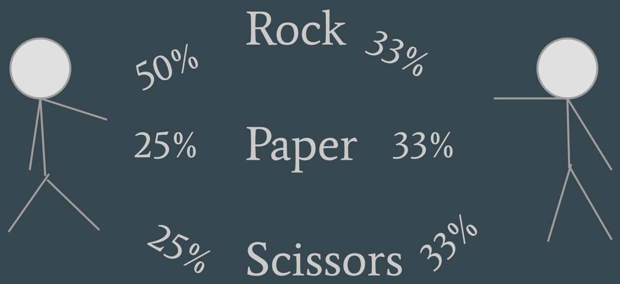

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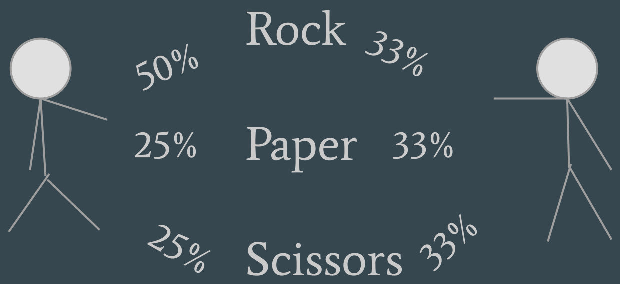
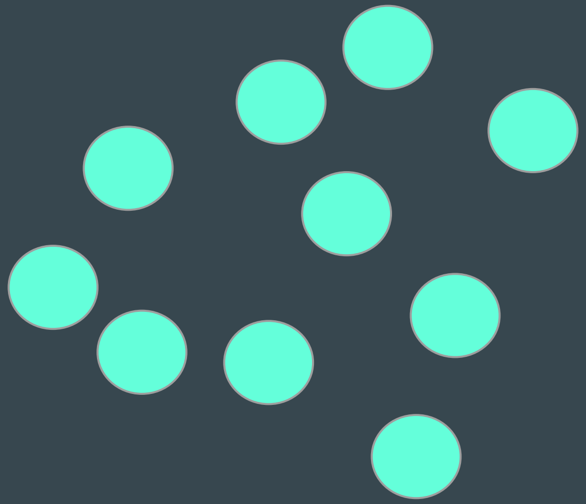


Is there now a Nash Equilibrium?

- Yes! Picking everything 33% → best response to itself!

Evolutionary Stable Strategy



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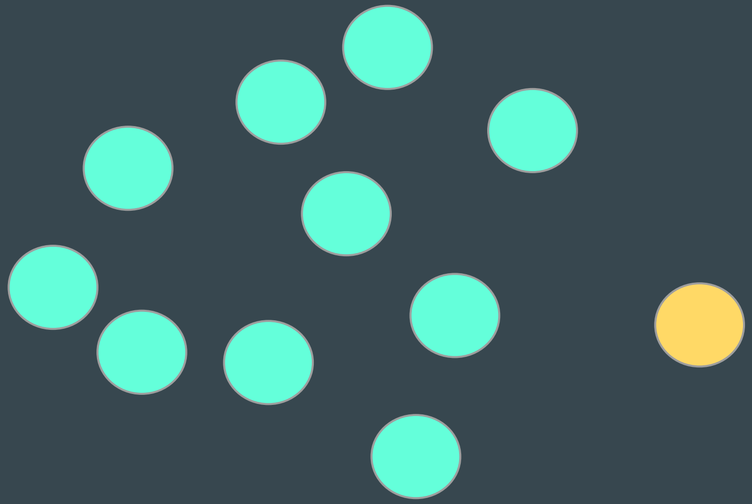
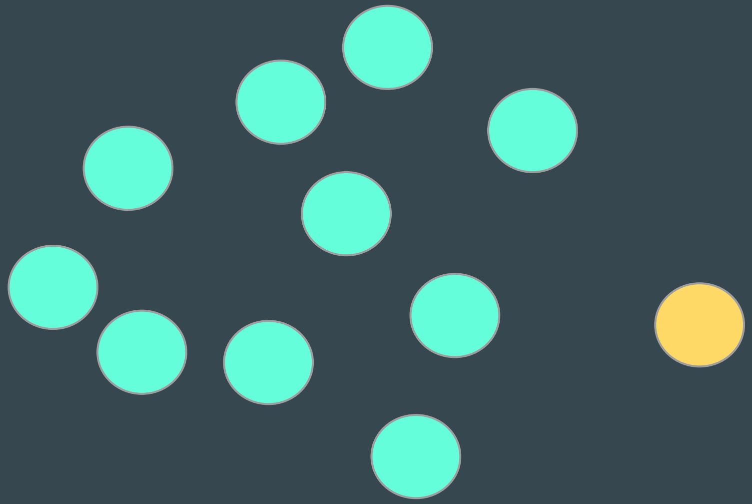


Evolutionary Stable Strategy

If yellow is worse against green...



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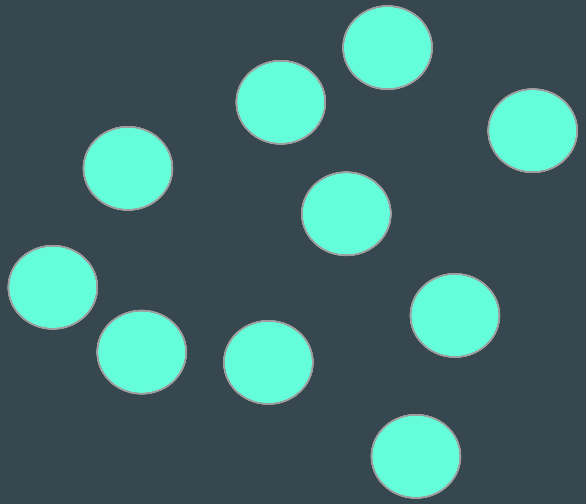
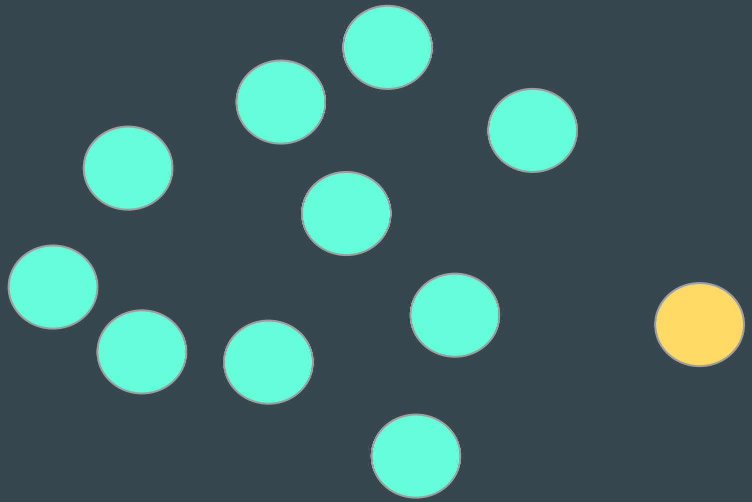


Evolutionary Stable Strategy

If yellow is as good against green...



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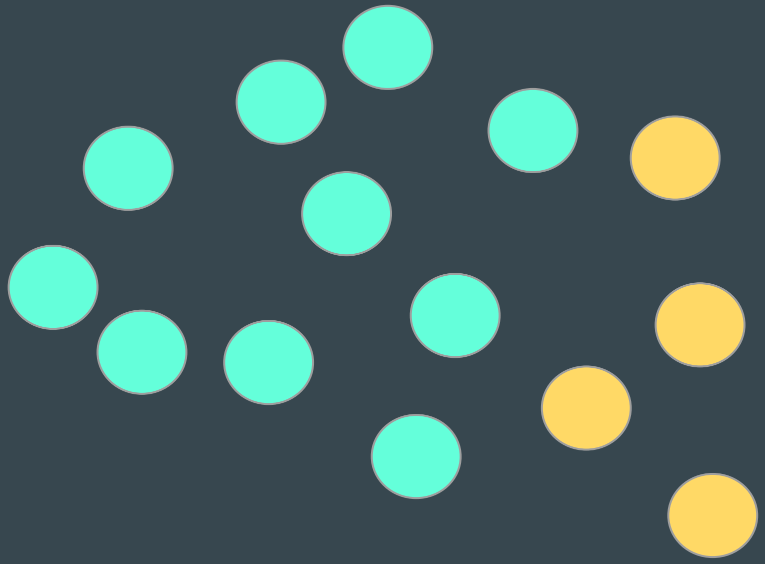
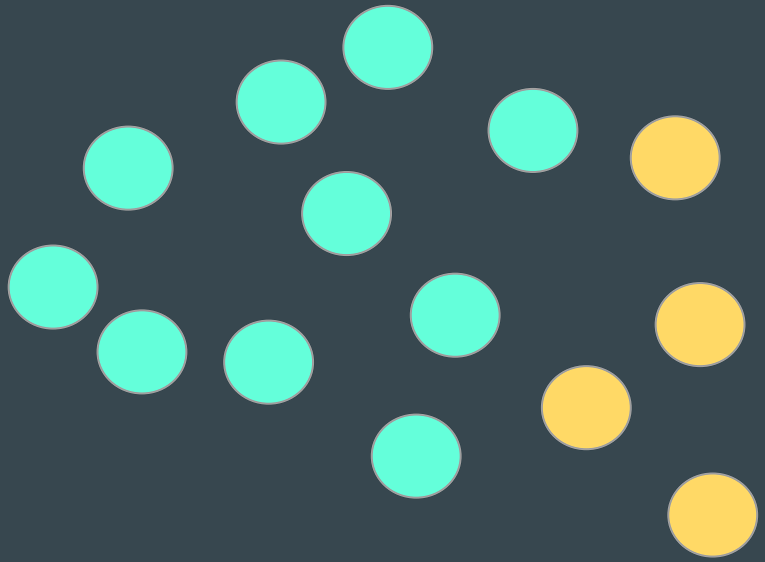


Evolutionary Stable Strategy

If yellow is as good against green... and is worse against itself!



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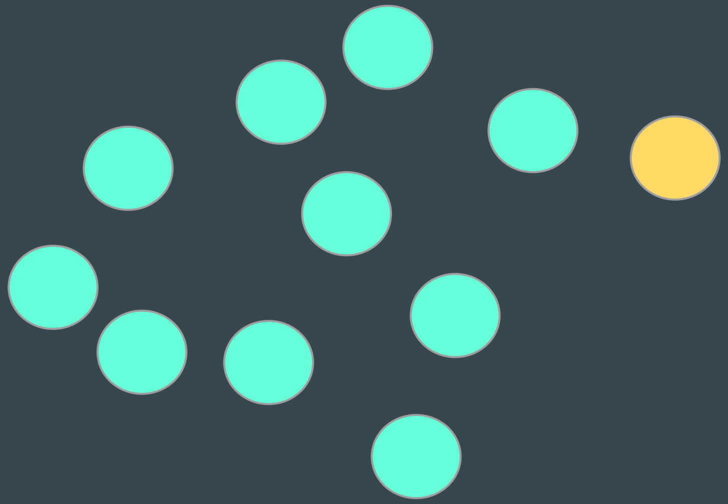

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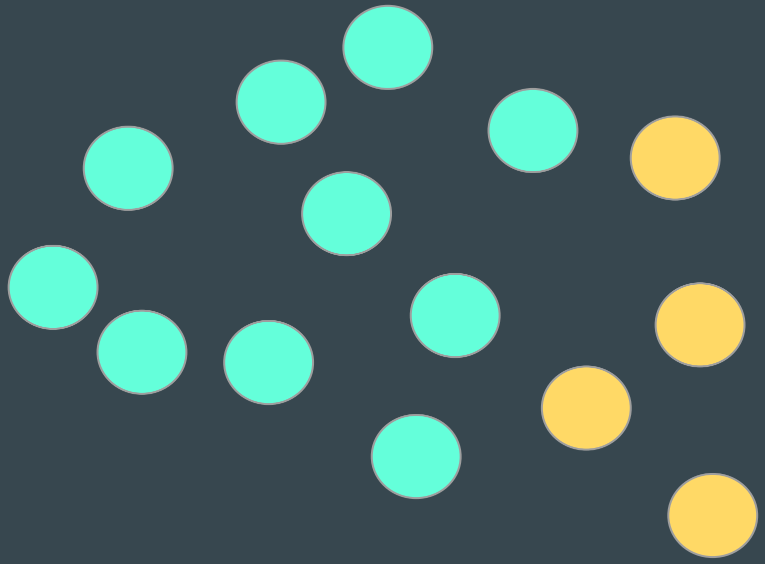

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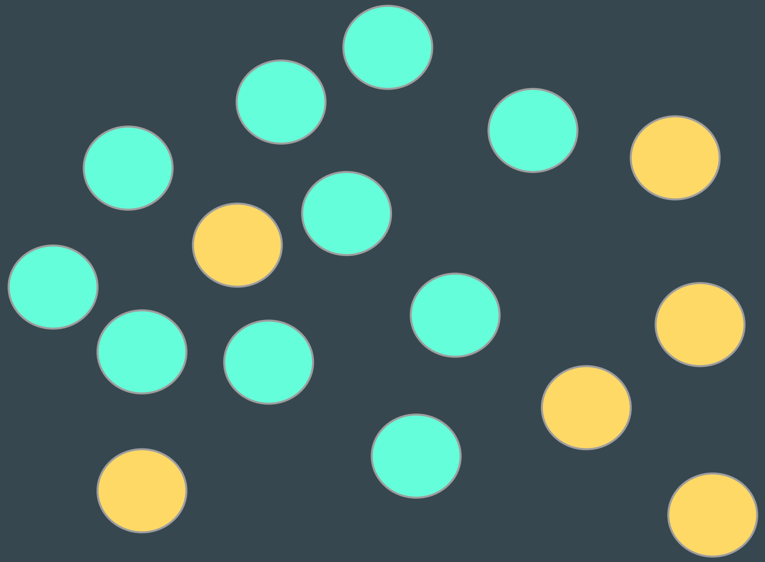

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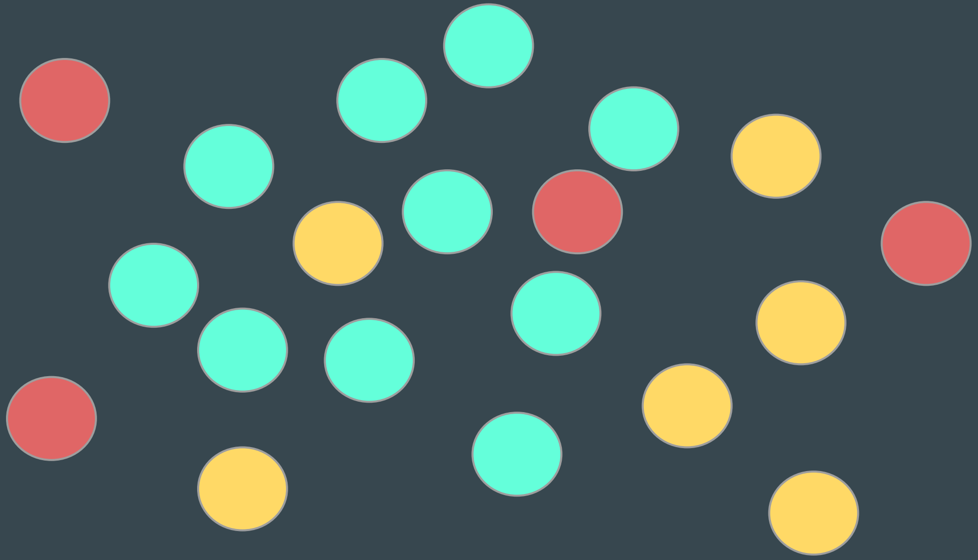


Evolutionary Stable Strategy

- Nash Equilibrium: *Strategy x is the best response to itself.*
- + extra condition: *For every Strategy y, such that y is an equally good response to x, it holds that y is a strictly worse response to itself, than x is to y.*

Rock-Paper-Scissors, evolutionary stable?

	Rock	Paper	Scissors
Rock	0/0	1/-1	-1/1
Paper	-1/1	0/0	1/-1
Scissors	1/-1	-1/1	0/0

Complexity

- Measure for how “hard” a problem is to solve
- How many steps does it take to complete a task, in relation to the input size?
- As the input size grows, how much longer does it take to solve a problem?

Complexity Classes

- Polynomial: Sorting a List
- Exponential time: Chess

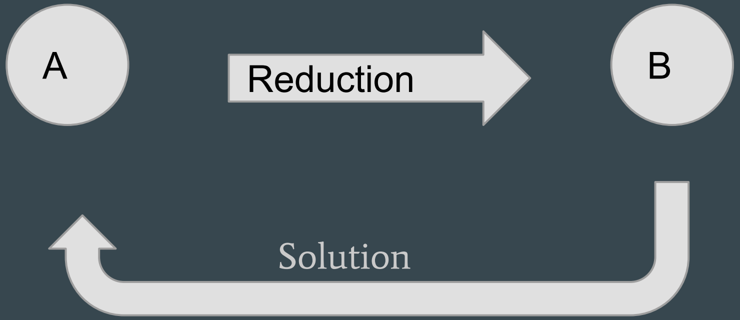
NP-Problems and co-NP-Problems

- NP: hard to find a solution, easy to check if the solution is correct
- Sudoku, Super Mario Bros, etc

- co-NP: similar but opposite of NP-problems
- In NP-Problems, yes-instances are easy to check, in co-NP-Problems no-instances are easy to check

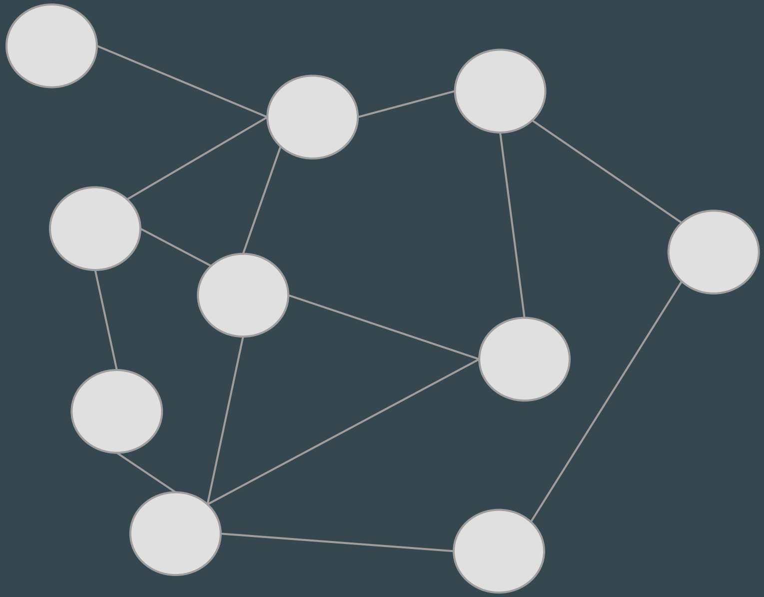
Reduction

- “quick” transformation of a problem A into another problem B, so that we can use a solution to problem B, to solve Problem A



Reduction, Example

Q: Is this graph three-colorable?



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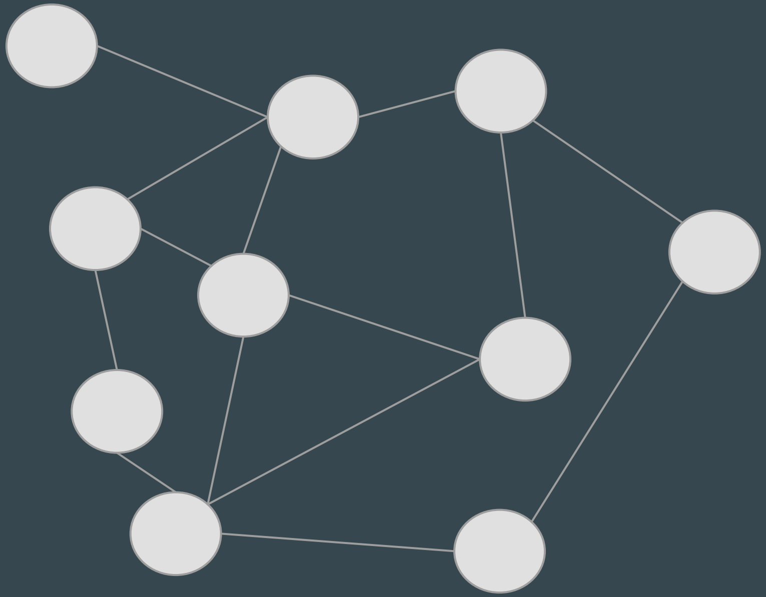
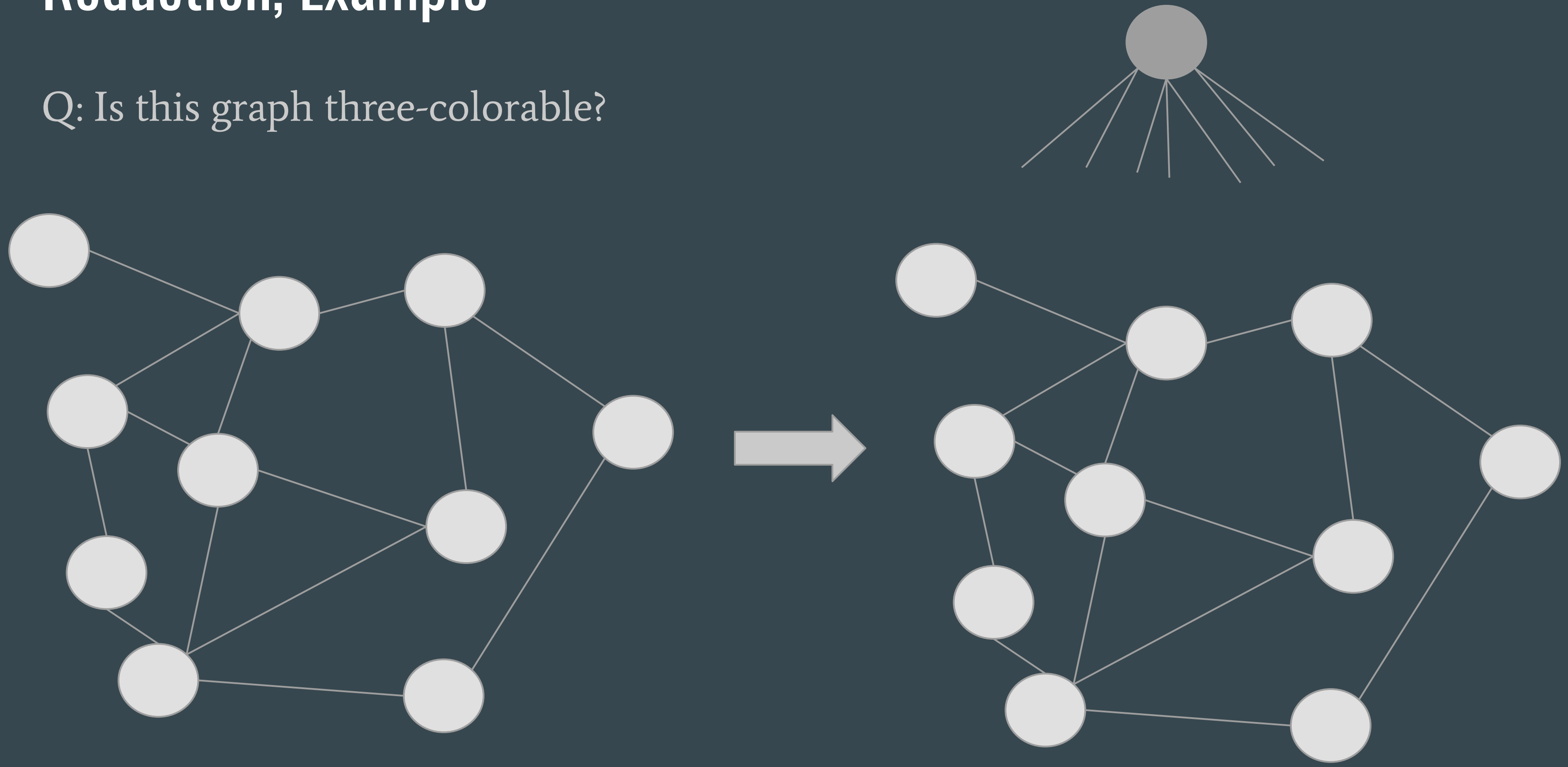


Idk, but I would know how to solve for four-colorability....

Reduction, Example

Q: Is this graph three-colorable?



What does this tell us about complexity?

- If there is a reduction from problem A to problem B, A is at most as hard as B.
- $\rightarrow$ B is as least as hard as A

NP-Hardness and co-NP-Hardness

- A problem *H* is NP-hard when for every problem *L* in NP, there is a polynomial-time reduction from *L* to *H*.
- Informally: "Hardest Problems in NP"

- A problem *H* is co-NP-hard when for every problem *L* in co-NP, there is a polynomial-time reduction from *L* to *H*

The Paper, finally, what is it about?

- Given Graph G, Integer k

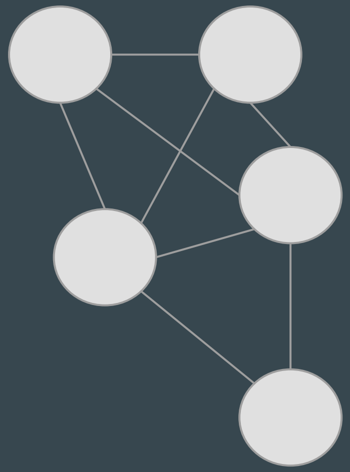
  → Payoff Matrix u

- u has ESS iff G has max clique size not exactly k

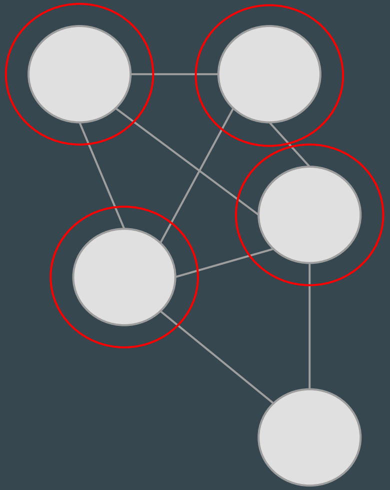
- finding max clique size is NP-hard and co-NP-hard

  → finding an ESS is NP-hard and co-NP-hard

Cliques



Cliques



Notation

			i	
0.1				
0.5				
0.4	j		u(i,j)	

- u: payoff matrix
- $u(i, j)$: payoff for option i when facing option j
- x, y: Strategies, probability distributions on options
- $u(x, y) = \Sigma_i \Sigma_j x_i * y_j * u(i, j)$: expected payoff of strategy x when facing y

- Symmetric Nash equilibrium: *for every y,* $u(x, x) \geq u(y, x)$
- 2. Condition: *for every* $y \neq x$ *such that* $u(y, x) = u(x, x)$, *we have that* $u(y, y) < u(x, y)$

The Reduction

Given: Graph G & Integer k, $1 < k <$ (number of vertices)

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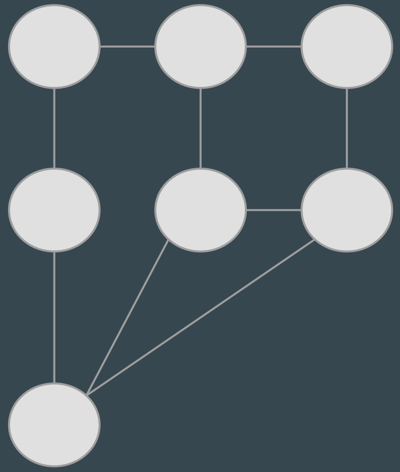
Given: Graph G & Integer k, $1 < k <$ (number of vertices), e.g. $k = 2$



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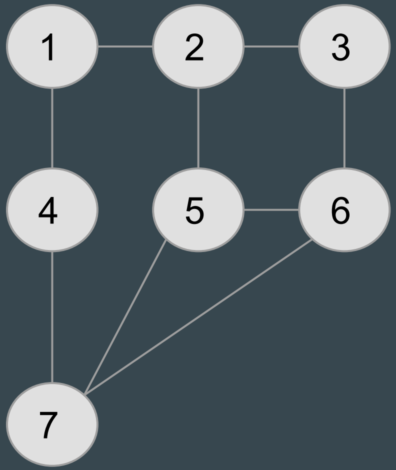

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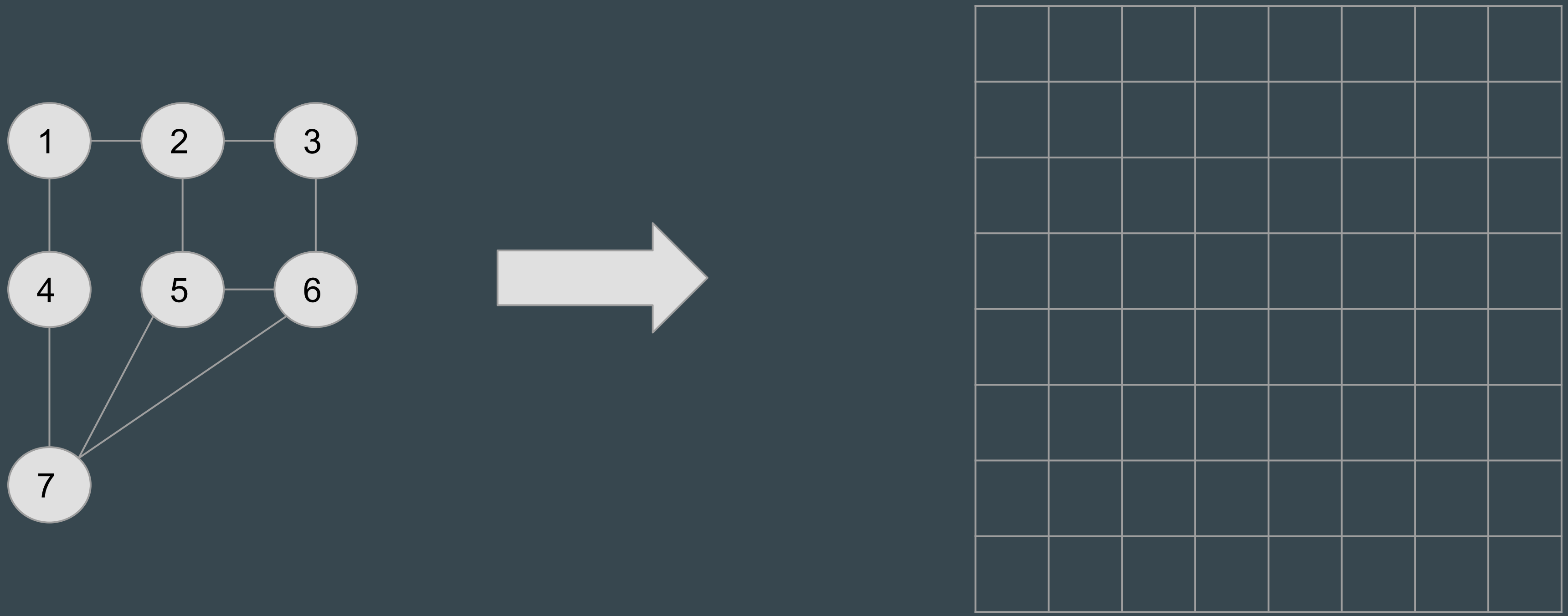
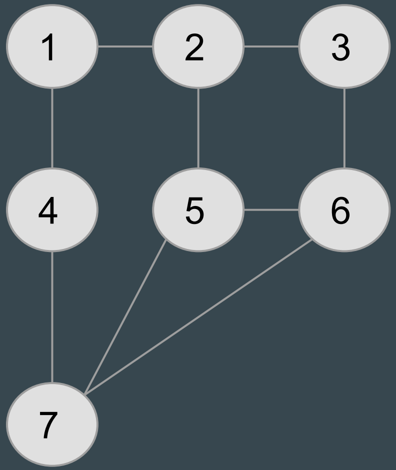


Rule 1

for $i, j > 0$, $i \neq j$, $u(i, j) = 1$, if there is an edge between vertices i & j, else 0

The Reduction

Given: Graph G & Integer k, $1 < k <$ (number of vertices), e.g. $k = 2$



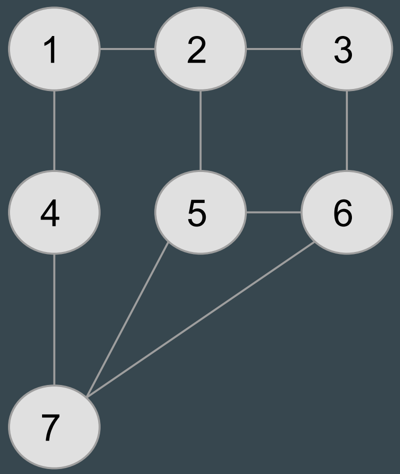
		1	0	1	0	0	0
	1		1	0	1	0	0
	0	1		0	0	1	0
	1	0	0		0	0	1
	0	1	0	0		1	1
	0	0	1	0	1		1
	0	0	0	1	1	1	

Rule 2

for $i > 0$, $u(i, i) = 0.5$

The Reduction

Given: Graph G & Integer k, $1 < k <$ (number of vertices), e.g. $k = 2$



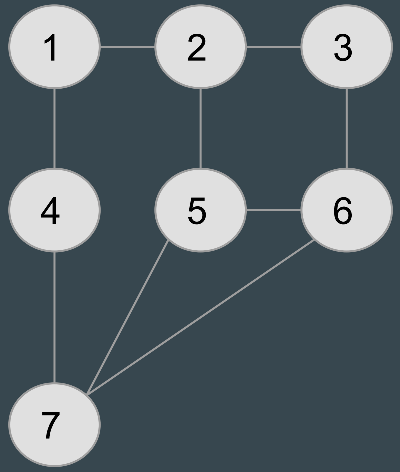
	.5	1	0	1	0	0	0
	1	.5	1	0	1	0	0
	0	1	.5	0	0	1	0
	1	0	0	.5	0	0	1
	0	1	0	0	.5	1	1
	0	0	1	0	1	.5	1
	0	0	0	1	1	1	.5

Rule 3

$u(i, 0) = u(0, i) = a = 1 - 1/(2k)$

The Reduction

Given: Graph G & Integer k, $1 < k <$ (number of vertices), e.g. $k = 2$, $a = 1 - 1/(2k) = 0.75$



a	a	a	a	a	a	a	a
a	.5	1	0	1	0	0	0
a	1	.5	1	0	1	0	0
a	0	1	.5	0	0	1	0
a	1	0	0	.5	0	0	1
a	0	1	0	0	.5	1	1
a	0	0	1	0	1	.5	1
a	0	0	0	1	1	1	.5

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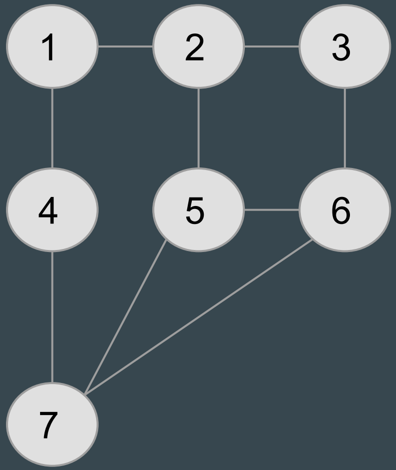
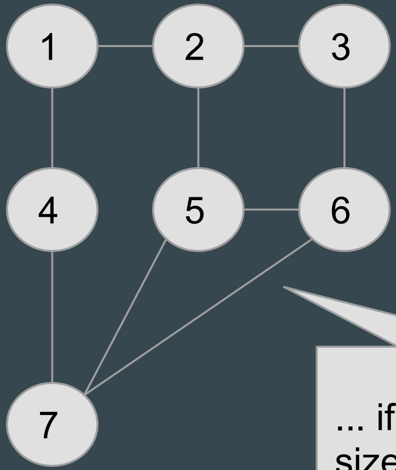


a	a	a	a	a	a	a	a
a	.5	1	0	1	0	0	0
a	1	.5	1	0	1	0	0
a	0	1	.5	0	0	1	0
a	1	0	0	.5	0	0	1
a	0	1	0	0	.5	1	1
a	0	0	1	0	1	.5	1
a	0	0	0	1	1	1	.5

What are we trying to show?

Has an evolutionary stable strategy...



... if in here the max clique size is not exactly k!

a	a	a	a	a	a	a	a
a	.5	1	0	1	0	0	0
a	1	.5	1	0	1	0	0
a	0	1	.5	0	0	1	0
a	1	0	0	.5	0	0	1
a	0	1	0	0	.5	1	1
a	0	0	1	0	1	.5	1
a	0	0	0	1	1	1	.5

Lemma

Lemma: For every x, with $x_0 = 0$, $u(x, x) \leq 1 - 1/(2k')$, where k' is the size of the maximum clique in G. Equality is achieved iff x is uniform over a k'-clique.

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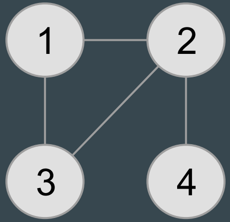

a	a	a	a	a
a	.5	1	1	0
a	1	.5	1	1
a	1	1	.5	0
a	0	1	0	.5

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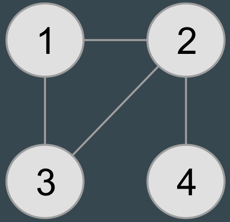

	0	0.3	0.3	0.3	0
0	a	a	a	a	a
0.3	a	.5	1	1	0
0.3	a	1	.5	1	1
0.3	a	1	1	.5	0
0	a	0	1	0	.5

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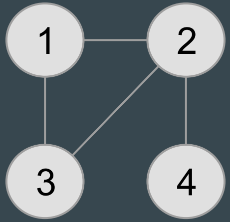

	0	0.3	0.3	0.3	0
0	a	a	a	a	a
0.3	a	.5	1	1	0
0.3	a	1	.5	1	1
0.3	a	1	1	.5	0
0	a	0	1	0	.5

If the support of x is a clique of size k'':

$$u(x, x) = 1 - \sum_i x_i^2/2$$

Lemma

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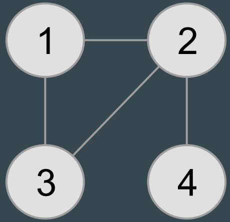

	0	0.3	0.3	0.3	0
0	a	a	a	a	a
0.3	a	.5	1	1	0
0.3	a	1	.5	1	1
0.3	a	1	1	.5	0
0	a	0	1	0	.5

If the support of x is a clique of size k'':

$$u(x, x) = 1 - \sum_i x_i^2/2 \leq 1 - 1/(2k'')$$

Lemma

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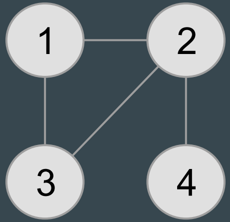

	0	0.2	0.3	0.4	0.1
0	a	a	a	a	a
0.2	a	.5	1	1	0
0.3	a	1	.5	1	1
0.4	a	1	1	.5	0
0.1	a	0	1	0	.5

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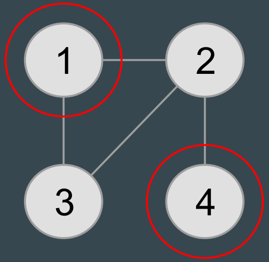

	0	0.2	0.3	0.4	0.1
0	a	a	a	a	a
0.2	a	.5	1	1	0
0.3	a	1	.5	1	1
0.4	a	1	1	.5	0
0.1	a	0	1	0	.5

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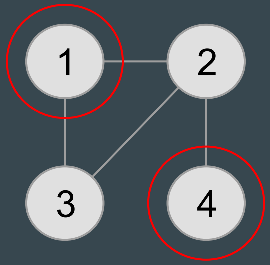

	0	0.2	0.3	0.4	0.1
0	a	a	a	a	a
0.2	a	.5	1	1	0
0.3	a	1	.5	1	1
0.4	a	1	1	.5	0
0.1	a	0	1	0	.5

p = sum of all x_i where 1 and i share an edge
= 0.3 + 0.4 = 0.7

q = sum of all x_i where 4 and i share an edge
= 0.3

Lemma

Lemma: For every x, with $x_0 = 0$, $u(x, x) \leq 1 - 1/(2k')$, where k' is the size of the maximum clique in G. Equality is achieved iff x is uniform over a k'-clique.

$x = (0, 0.2, 0.3, 0.4, 0.1)$, $x' = (0, 0.3, 0.3, 0.4, 0)$

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$x = (0, 0.2, 0.3, 0.4, 0.1)$, $x' = (0, 0.3, 0.3, 0.4, 0)$

$u(x', x') = \mathrm{u}(x, x) + x_4(p - q) + x_4 x_1$

The Claims

Claim 1: If C is a maximal clique of G of size $k' > k$, and x is the uniform distribution on C, then x is an ESS.

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$u(x, x) = 1 - 1/(2k')$

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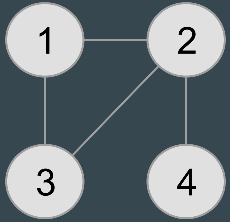
$u(x, x) = 1 - 1/(2k')$, $u(0, x) = a = 1 - 1/(2k)$

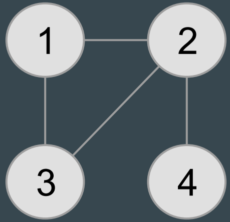


	0	0.3	0.3	0.3	0
1	a	a	a	a	a
0	a	.5	1	1	0
0	a	1	.5	1	1
0	a	1	1	.5	0
0	a	0	1	0	.5

The Claims

Claim 1: If C is a maximal clique of G of size $k' > k$, and x is the uniform distribution on C, then x is an ESS.

$u(x, x) = 1 - 1/(2k')$, $u(0, x) = a = 1 - 1/(2k)$, $u(i, x) < u(x, x) \;\; i \notin C$



	0	0.3	0.3	0.3	0
0	a	a	a	a	a
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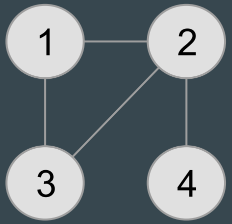

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- Claim 2: If *G* contains no clique of size *k* then the pure strategy 0 is an ESS.

Summary

- How to get payoff matrix from graph G and integer k
- Matrix has ESS iff G has not max clique size k
- The Problem of having max clique size k is NP-hard and co-NP-hard

$\Rightarrow$ finding an ESS is also NP-hard and co-NP-hard

Questions?

Thanks for your Attention!