Using flies to find algorithms

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A Biological Solution to a Fundamental Distributed Computing Problem by Afek et. al.

Algorithm

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Algorithm | Approach

Outline

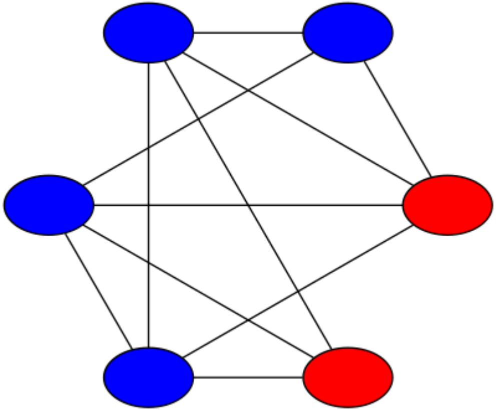
1. Problem
2. Approach
3. Solution
4. Critique

Outline

Problem

Maximal Independent Set (MIS)

Example



Example

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Maximal Independent Set (MIS)

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Maximal Independent Set (MIS)

Given a graph $G = (V, E)$, find $A \subseteq V$, s.t.

- No two $v \in A$ are connected – **independence**
- Adding any $v \notin A$ to A violates previous property – **maximality**

Maximal Independent Set (MIS)
Equivalent definition

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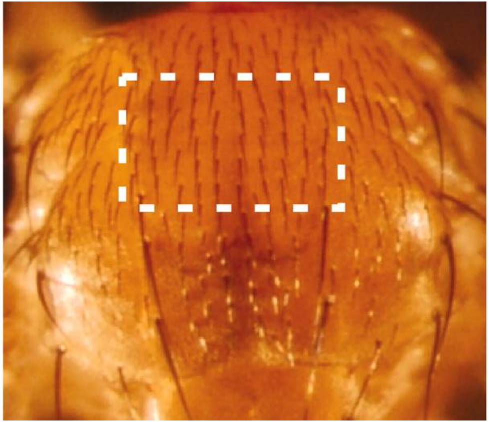
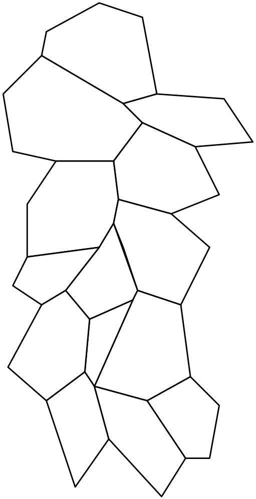
- No two nodes in A are **adjacent to each other**

Outline



Example



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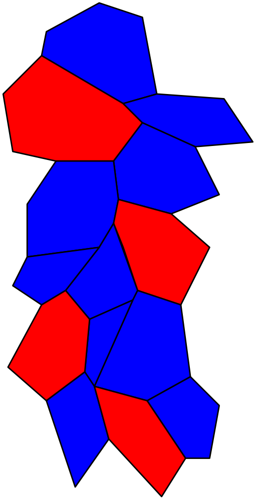

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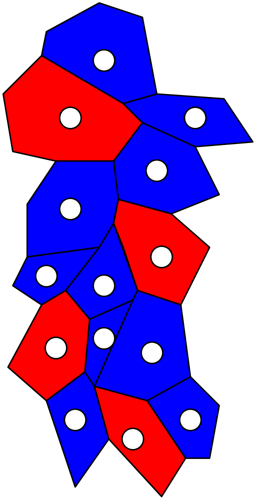
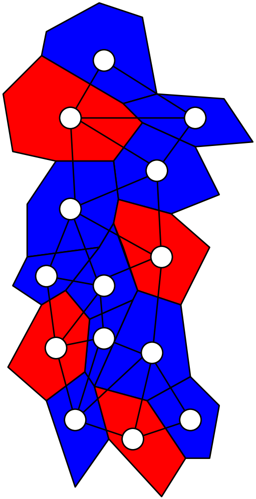


Example



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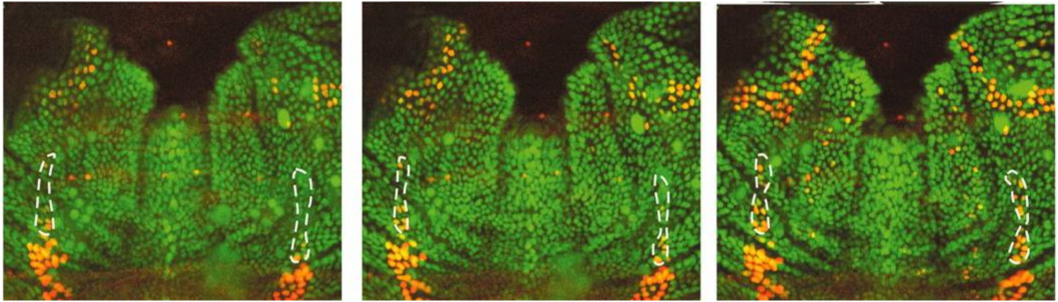
- Algorithm **runs on nodes**
- Nodes can **communicate via edges**
- Nodes have **no initial knowledge** of topology of graph
- Nodes can’t **selectively recevie or send messages**



Outline

Algorithm

1. With probability p: Broadcast
2. If no other node broadcasts:
   - Exit as MIS
3. If neighbour exits as MIS:
   - Exit as non-MIS

Otherwise: Fail.

Proofs

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Proofs

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- **Probability of Success** - $1 - \frac{\log n}{n^2}$
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Success = No neighours

To bound probability: Consider **maximum degree** D of graph.

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Maximum degree **halves every round** with high probability.

After $\log D$ rounds...

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$$\frac{D}{2^{\log D}} = 1$$

Consider v with "too high" degree for round i

Probability v **is removed** is:

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Probability v **is removed** is:

- v exits as MIS
- neighbour of v exits as MIS

i.e.

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$$P(v \text{ or neighbor of } v \text{ broadcasts}) \cdot P(\text{broadcasting node is sole broadcaster})$$

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turns out:

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turns out: **constant** for their choice of broadcasting probability.

Repeat this step $O(\log n)$ often...

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$P(\text{success})$

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- ...adaption of previous solution to constraints of fly?

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  Cited paper:

$$P(...) \geq (1 - \frac{1}{d+1})^d > \frac{1}{e}$$

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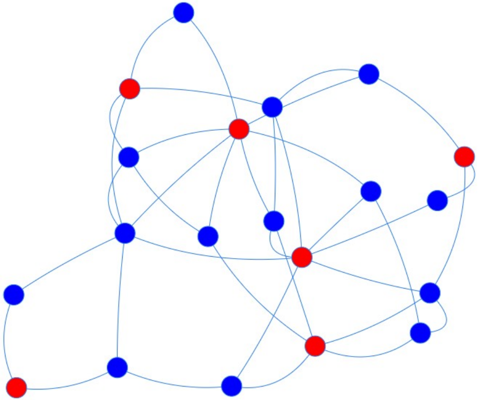
  Error: $>$ **0.3** for $d = 1$



Questions?

Show that it halves in rounds $0 \ldots \log D$:

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$$P(\text{halved in rounds } 0..i)$$
$$\geq P(\text{halved in rounds } 0..i \mid \text{halved in rounds } 0...i-1)$$
$$\cdot P(\text{halved in rounds } 0...i-1)$$

Lemma 4

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$$P(D_i \leq \frac{D}{2^i})$$

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- $D =$ max degree of graph + 1
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$$P(D_i \leq \frac{D}{2^i}) \geq 1 - \frac{i}{n^2}$$

Theorem 1

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$$
\begin{aligned}
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Lemma 4

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A_i = in round i, degree is at most $\frac{D}{2^i}$

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$$P(A_i) \geq 1 - \frac{i}{n^2}$$

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$$Q(i) := P(A_i) \geq p(i)$$

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$$Q(0) \land Q(i-1) \rightarrow Q(i)$$

Lemma 4

$$P(A_i)$$

Lemma 4

$$P(A_i) = P(A_i|A_{i-1})P(A_{i-1}) + P(A_i|\neg A_{i-1})P(\neg A_{i-1})$$

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Frame Title

$$P(A_i|A_{i-1})$$

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"after round i, max degree is $d := \frac{D}{2^i}$, assuming before round it's $\frac{D}{2^{i-1}} = 2d$"

Frame Title

d max degree after round i
$2d$ max degree before round i

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$\frac{1}{d}$ probability for node to broadcast in round i

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Let $v =$ node with degree $> d$.

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$$P(v \text{ or neighbour of } v \text{ broadcasts})$$

Frame Title

$$\begin{array}{rl} d & \text{max degree after round } i \\ 2d & \text{max degree before round } i \\ \frac{1}{d} & \text{probability for node to broadcast in round } i \end{array}$$

Let $v =$ node with degree $> d$.Then

$$\begin{aligned} & P(v \text{ or neighbour of } v \text{ broadcasts}) \\ = \quad & 1 - P(\text{neither } v \text{ nor neighbour of } v \text{ broadcasts}) \end{aligned}$$

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$\frac{1}{d}$ probability for node to broadcast in round i

$$P(\text{broadcast doesn't collide}|\text{broadcast happens})$$

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d max degree after round i
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$$\begin{aligned} & P(\text{broadcast doesn't collide}|\text{broadcast happens}) \\ \geq \quad & (1 - P(\text{neighbour broadcasts}))^{2d} \end{aligned}$$

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Hence, probability that v is removed in this step is is

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$$\geq \quad (1 - \frac{1}{e})\frac{1}{e^2}$$

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Hence, probability that v is removed in this step is is

$$\begin{aligned} &\geq (1 - \frac{1}{e})\frac{1}{e^2} \\ &> \frac{1}{2^4} \end{aligned}$$

$P(v \text{ removed this round})$

$P(v \text{ removed this round})) \geq ...$

$$P(v \text{ removed this round})) \geq ... > 1 - \frac{1}{n^3}$$

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$\Rightarrow$

$$P(v \text{ removed this round}) \geq ... > 1 - \frac{1}{n^3}$$

$$\Rightarrow \quad P(\text{all nodes with deg} > d \text{ removed})$$

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$$\Rightarrow \quad P(\text{all nodes with deg} > d \text{ removed}) \geq 1 - \frac{1}{n^2}$$
$$= P(A_i | A_{i-1})$$

Putting it together

$$P(A_i)$$

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$$P(A_i) \quad \geq \quad P(A_i|A_{i-1})P(A_{i-1})$$

Putting it together

$$\begin{aligned} P(A_i) &\geq P(A_i|A_{i-1})P(A_{i-1}) \\ &\geq (1 - \frac{1}{n^2})P(A_{i-1}) \end{aligned}$$

Putting it together

$$\begin{aligned} P(A_i) &\geq P(A_i|A_{i-1})P(A_{i-1}) \\ &\geq (1 - \frac{1}{n^2})P(A_{i-1}) \\ &\geq (q - \frac{1}{n^2})(1 - \frac{i-1}{n^2}) \end{aligned}$$

Putting it together

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&> 1 - \frac{i}{n^2} \\
&\equiv \text{Lemma 4}
\end{aligned}
$$